

GEOMETRIC INTERSECTION NUMBER OF SIMPLE CLOSED CURVES ON A SURFACE AND SYMPLECTIC EXPANSIONS OF FREE GROUPS

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ABSTRACT. For two oriented simple closed curves on a compact orientable surface with a connected boundary we introduce a simple computation of a value in the first homology group of the surface, which detects in some cases that the geometric intersection number of the curves is greater than zero when their algebraic intersection number is zero. The value, computed from two elements of the fundamental group of the surface corresponding to the curves, is found in the difference between one of the elements and its image of the action of Dehn twist along the other. To give a description of the difference symplectic expansions of free groups is an effective tool, since we have an explicit formula for the action of Dehn twist on the target space of the expansion due to N. Kawazumi and Y. Kuno.

1. INTRODUCTION

Let $\Sigma = \Sigma_{g,1}$ be a compact orientable surface of genus $g \geq 1$ with a connected boundary. Let α and β be oriented simple closed curves on Σ . We always assume that the intersections of the curves are transverse double points. The geometric intersection number of α with β , we denote by $i_G(\alpha, \beta)$, is the minimal number of intersection points of α with any simple closed curve on Σ freely homotopic to β . The algebraic intersection number of α with β , we denote by $i_A(\alpha, \beta)$, is the sum of signs of intersection points of α with β , where the sign of an intersection point of α with β is $+1$ when a pair of tangent vectors of α and β in this order is consistent with an oriented basis for Σ , otherwise the sign is -1 .

In this paper we describe an algebraic value associating with two simple closed curves α and β on Σ , which may detect that $i_G(\alpha, \beta)$ is greater than zero when $i_A(\alpha, \beta)$ is zero. Our strategy is as follows: The geometric intersection number of two simple closed curves on a surface is closely related to the change of the homotopy type of one of the curves by performing a Dehn twist along the other. In particular it is known (e.g. [1]) that $i_G(\alpha, \beta) = 0$ if and only if the Dehn twist along one of the curves does not change homotopy type of the other, in other words, there exists a based homotopy class b of a based loop freely homotopic to β such that $t_\alpha(b) = b$. To analyze the action of Dehn twist on the fundamental group of a surface, the symplectic expansion, a certain type of (generalized) Magnus expansion defined by G. Massuyeau [5], of free groups is an effective tool, since we have an explicit formula describing the action of Dehn twist on the target space of the expansion due to N. Kawazumi and Y. Kuno. (See §2 for detail.) Applying the formula we can compute differences between expansion of $t_\alpha(b)$ and of b . Since the first degree part of the expansions of $t_\alpha(b)$ and b are coincide because of the condition $i_A(\alpha, \beta) = 0$, we will focus on the second degree part of them.

In the rest of this section we mention our main theorem and demonstrate an application of the theorem. In §2, we recall the terminology of Kawazumi-Kuno's theory developed in [3], and prepare some propositions for use in §3, in which we give a proof of the main theorem.

1.1. Main result. The fundamental group $\pi = \pi_1(\Sigma, p)$ of Σ with a point p on $\partial\Sigma$ is a free group of rank $2g$ and the first homology group of Σ with coefficients in \mathbb{Q} , $H = H_1(\Sigma, \mathbb{Q}) = \pi/[\pi, \pi] \otimes \mathbb{Q}$, is a free abelian group of rank $2g$. For $x \in \pi$, we denote by $|x|$ the element of H corresponding to x via the abelianization of π .

Let $\{x_1, y_1, \dots, x_g, y_g\}$ be a symplectic generators of π as in Figure 1. Putting

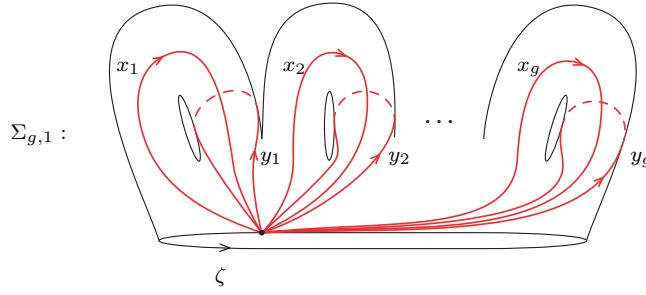


FIGURE 1. symplectic generators of π .

$X_i = |x_i|$, $Y_i = |y_i|$ ($i = 1, 2, \dots, g$), we have a symplectic basis $\{X_1, Y_1, \dots, X_g, Y_g\}$ of H , namely the basis satisfies $X_i \cdot Y_i = \delta_{ij}$, $X_i \cdot X_j = Y_i \cdot Y_j = 0$ for $1 \leq i, j \leq g$, where $\cdot : H \times H \rightarrow \mathbb{Q}$ is the intersection form on H .

With respect to the symplectic generators of π , we define a map $\ell : \pi \rightarrow H \wedge H$ by the following three rules:

- i) $\ell(1) = 0$,
- ii) $\ell(x_i) = \frac{1}{2}X_i \wedge Y_i$, $\ell(y_i) = -\frac{1}{2}X_i \wedge Y_i$, ($i = 1, 2, \dots, g$),
- iii) For $\forall g, h \in \pi$, $\ell(gh) = \ell(g) + \ell(h) + \frac{1}{2}|g| \wedge |h|$.

Note that we obtain $\ell(g^{-1}) = -\ell(g)$ for $\forall g \in \pi$ from i) and iii).

We consider that $H \wedge H$ acts on H as follows: For $X \wedge Y \in H \wedge H$ and $Z \in H$,

$$(X \wedge Y)(Z) := (Z \cdot X)Y - (Z \cdot Y)X.$$

Theorem 1.1. *Let α and β be oriented simple closed curves on Σ with $i_A(\alpha, \beta) = 0$. If $\ell(a)(|b|) + \ell(b)(|a|)$ is not an element in $\mathbb{Z}|a| + \mathbb{Z}|b| \subset H$, where a and b are based homotopy classes of based loops freely homotopic to α and β respectively, then $i_G(\alpha, \beta) > 0$.*

Remark 1.1. (1) When both of the curves are separating on Σ , the theorem is trivial since $\ell(a)(|b|) + \ell(b)(|a|) = 0 \in \mathbb{Z}|a| + \mathbb{Z}|b|$.
 (2) In the case where both of the curves are non-separating on Σ , if $|a|$ and $|b|$ are linearly dependent on H , the theorem is also trivial, namely it is always true that $\ell(a)(|b|) + \ell(b)(|a|) \in \mathbb{Z}|a| + \mathbb{Z}|b|$. See Theorem 3.3 in §3.

1.2. Example of computations in Theorem 1.1. Let α and β be oriented simple closed curves on $\Sigma_{2,1}$ as shown in the left part of Figure 2. Taking symplectic generators of π as in the right of the figure, we may choose elements $a = x_1 x_2 y_2 x_2^{-1}$ and $b = y_2 x_1^{-1}$ in π , which are based homotopy classes of based loops freely

homotopic to α and β respectively. We can confirm that the homology classes $|a| = X_1 + Y_2$ and $|b| = -X_1 + Y_2$ meet $|a| \cdot |b| = 0$.

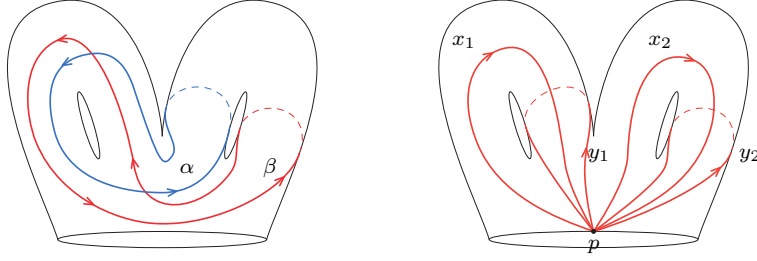


FIGURE 2. Oriented curves on $\Sigma_{2,1}$ and symplectic generators of π .

We now compute $\ell(a)$ and $\ell(b)$. In addition to the property of the map ℓ mentioned above, we will use the following.

$$\ell(ghg^{-1}) = \ell(h) + |g| \wedge |h|, \quad \forall g, h \in \pi.$$

(See Proposition 2.6 for a proof).

$$\begin{aligned} \ell(a) &= \ell(x_1) + \ell(x_2 y_2 x_2^{-1}) + \frac{1}{2} X_1 \wedge Y_2 \\ &= \frac{1}{2} (X_1 \wedge Y_1 + X_2 \wedge Y_2 + X_1 \wedge Y_2), \\ \ell(b) &= \ell(y_2) - \ell(x_1) - \frac{1}{2} Y_2 \wedge X_1 \\ &= \frac{1}{2} (-X_2 \wedge Y_2 - X_1 \wedge Y_1 - Y_2 \wedge X_1). \end{aligned}$$

Elements $\ell(a)$ and $\ell(b)$ in $H \wedge H$ act on $|b|$ and $|a|$ respectively as follows;

$$\begin{aligned} \ell(a)|b| &= \frac{1}{2} (X_1 \wedge Y_1 + X_2 \wedge Y_2 + X_1 \wedge Y_2) (-X_1 + Y_2) = \frac{1}{2} (X_1 - Y_2), \\ \ell(b)|a| &= \frac{1}{2} (-X_2 \wedge Y_2 - X_1 \wedge Y_1 - Y_2 \wedge X_1) (X_1 + Y_2) = \frac{1}{2} (X_1 + Y_2). \end{aligned}$$

Now we have that

$$\ell(a)|b| + \ell(b)|a| = X_1 = \frac{|a| - |b|}{2} \notin \mathbb{Z}|a| + \mathbb{Z}|b|,$$

and see from Theorem 1.1 that $i_G(\alpha, \beta) > 0$.

Remark 1.2. The converse of Theorem 1.1 does not hold. The following is a counter-example: We set $\Sigma = \Sigma_{2,1}$. The symplectic generators of π as above. Let α be an oriented simple closed curve on Σ freely homotopic to $a = x_1$. Take a word $b' = x_2^{-1} \in \pi$. It is easy to see that $\ell(a)|b'| + \ell(b')|a| = 0$. We now consider a word $b = b'[y_1, \zeta]\zeta$, where $\zeta = [x_1, y_1][x_2, y_2]$, the homotopy class of the boundary loop. We can confirm that b has a representative of a simple loop, say β , and $i_G(\alpha, \beta) = 2$ by the bigon criterion. Setting $\omega := \ell(\zeta) = X_1 \wedge Y_1 + X_2 \wedge Y_2$, we have that $\ell(b) = \ell(b') + \omega$ (see Proposition 2.6). From facts that $|b| = |b'|$ and $\omega(|a|) = -|a| \in \mathbb{Z}|a|$ (see Proposition 2.8), we obtain

$$\ell(a)|b| + \ell(b)|a| = \ell(a)|b'| + \ell(b')|a| + \omega(|a|) = -|a| \in \mathbb{Z}|a| \subset \mathbb{Z}|a| + \mathbb{Z}|b|.$$

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2. PRELIMINARIES

In this section we recall some of notions in the theory about the mapping class group of a surface developed in [2], [5] and [3], and prepare propositions, most of all are with a proof, for use in §3.

We denote by \widehat{T} the completed tensor algebra generated by H , i.e., $\widehat{T} = \prod_{m=0}^{\infty} H^{\otimes m}$. We will write the tensor product of u and v in \widehat{T} as uv omitting the tensor symbol. We set $\widehat{T}_p = \prod_{m \geq p}^{\infty} H^{\otimes m}$ for $p \geq 1$, which gives a filtration of \widehat{T} .

2.1. Magnus expansion of π . The subset $1 + \widehat{T}_1$ of \widehat{T} has a group structure by the tensor product of \widehat{T} .

Definition 2.1 (Kawazumi, [2]). A Magnus expansion θ of π is a group homomorphism from π to $1 + \widehat{T}_1$ satisfying

$$\theta(x) \equiv 1 + |x| \pmod{\widehat{T}_2} \text{ for } \forall x \in \pi.$$

For a Magnus expansion θ , we set $\ell^\theta = \log \circ \theta$, where the logarithm map is defined as

$$\log(u) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (u - 1)^k \text{ for } \forall u \in 1 + \widehat{T}_1.$$

We denote by $\widehat{\mathcal{L}}$ the completed free Lie algebra generated by H with the bracket $[u, v] = uv - vu$ and set $\mathcal{L}_p = \widehat{\mathcal{L}} \cap H^{\otimes p}$ for $p \geq 1$.

Definition 2.2. A group-like expansion of π is a Magnus expansion θ of π satisfying

$$\ell^\theta(\pi) \subset \widehat{\mathcal{L}}.$$

We take symplectic generators $\{x_j, y_j\}_{1 \leq j \leq g}$ of π as shown in Figure 1. Let ζ be a homotopy class of the boundary loop of Σ as in the same figure, which is expressed in the word of the symplectic generators as $\zeta = \prod_{j=1}^g [x_j, y_j]$. Set $\omega = \sum_{j=1}^g [X_j, Y_j] \in \mathcal{L}_2$, where $X_j = |x_j|$ and $Y_j = |y_j|$ ($j = 1, 2, \dots, g$).

Definition 2.3 (Massuyeau, [5]). A symplectic expansion of π is a group-like expansion θ of π satisfying

$$\ell^\theta(\zeta) = \omega.$$

2.2. The logarithm of Dehn twists. Let $\text{Aut}(\widehat{T})$ be the group of the filter-preserving algebra automorphisms of \widehat{T} , and $\text{Der}(\widehat{T})$ the space of the derivations of \widehat{T} . We may identify \widehat{T}_1 with $\text{Der}(\widehat{T})$ as follows:

By the Poincaré duality $\widehat{T}_1 = H \otimes \widehat{T}$ is identified with $\text{Hom}(H, \widehat{T})$, namely, for $X_1 X_2 \cdots X_k \in H^{\otimes k}$ and $Y \in H$ we define

$$(2.1) \quad (X_1 X_2 \cdots X_k)(Y) = (Y \cdot X_1) X_2 \cdots X_k.$$

Since each $h \in \text{Hom}(H, \widehat{T})$ determines a derivation $D_h \in \text{Der}(\widehat{T})$ by

$$D_h(Y_1 Y_2 \cdots Y_p) = h(Y_1) Y_2 \cdots Y_p + Y_1 h(Y_2) \cdots Y_p + \cdots + Y_1 Y_2 \cdots h(Y_p)$$

and vice versa, we have an identification of \widehat{T}_1 with $\text{Der}(\widehat{T})$.

We denote by $\text{Aut}(\pi)$ the automorphism group of π . For $\phi \in \text{Aut}(\pi)$ An automorphism $T^\theta(\phi) \in \text{Aut}(\widehat{T})$ is called the total Johnson map of ϕ associated to a Magnus expansion θ , if it satisfies

$$T^\theta(\phi) \circ \theta = \theta \circ \phi.$$

It is known [2, §2.5] that there uniquely exists a total Johnson map for each $\phi \in \text{Aut}(\pi)$.

We denote by t_α a right-handed Dehn twist along a simple closed curve α on Σ . In [3], N. Kawazumi and Y. Kuno gave an explicit description of the total Johnson map of t_α with respect to a symplectic expansion θ as follows:

Let ν be a linear map on \widehat{T} , called the cyclic permutation, defined by

$$\nu(X_1 X_2 \cdots X_k) = X_2 \cdots X_k X_1 \quad (X_j \in H, j = 1, 2, \dots, k),$$

and N a linear map on \widehat{T} defined by

$$N|_{H^{\otimes k}} = \sum_{j=0}^{k-1} \nu^j,$$

for $k \geq 1$, and $N|_{\mathbb{Q}} = 0$. Kawazumi and Kuno [3, §2.6] introduced a map $L^\theta : \pi \rightarrow \widehat{T}_2$ with respect to a Magnus expansion θ defined by

$$L^\theta(x) = \frac{1}{2} N(\ell^\theta(x) \ell^\theta(x)), \quad [x \in \pi],$$

and they showed that $L^\theta(x^{-1}) = L^\theta(x)$ and $L^\theta(yxy^{-1}) = L^\theta(x)$ for any $x, y \in \pi$.

Let α be a simple closed curve on Σ . The value $L^\theta(a) \in \widehat{T}_2$ is well-defined because of the property of L^θ mentioned above. With respect to a symplectic expansion θ , an explicit description of the total Johnson map of t_a in the view of the identification $\widehat{T}_1 = \text{Der}(\widehat{T})$ is given [3, Theorem 1.1.1] as

$$(2.2) \quad T^\theta(t_a) = e^{-L^\theta(a)},$$

namely, for $\forall x \in \pi$

$$\theta(t_a(x)) = e^{-L^\theta(a)}(\theta(x)),$$

where the exponential map on \widehat{T} is defined as $e^u := \sum_{k=0}^{\infty} \frac{u^k}{k!}$ for $u \in \widehat{T}$.

2.3. Degree 2-part and 3-part of $L^\theta(x)$. In $H^{\otimes 2}$, we will identify \mathcal{L}_2 with $H \wedge H$ by identifying $[X, Y] = XY - YX$ with $X \wedge Y$ for $X, Y \in H$. In $H^{\otimes 3}$, we will regard $\wedge^3 H$ as a subspace of $H^{\otimes 3}$ by identifying $X \wedge Y \wedge Z$ with $XYZ + YZX + ZXY - XZY - ZYX - YXZ$. Thus an action of $H \wedge H$ and of $\wedge^3 H$ on H are given as follows.

Proposition 2.1. For $X, Y, Z \in H$ and $u \in H \wedge H = \mathcal{L}_2$,

- (1) $(X \wedge Y)(Z) = (Z \cdot X)Y - (Z \cdot Y)X$,
- (2) $(X \wedge u)(Z) = (Z \cdot X)u - X \wedge (u(Z))$.

Proof. (1) Computing the action of $X \wedge Y = [X, Y] = XY - YX$ on H as (2.1), we have

$$(X \wedge Y)(Z) = (XY)(Z) - (YX)(Z) = (Z \cdot X)Y - (Z \cdot Y)X.$$

(2) It is enough to show the equation for $u = A \wedge B$, $[A, B \in H]$.

$$\begin{aligned} (X \wedge (A \wedge B))(Z) &= (XAB + ABX + BXA - XBA - BAX - AXB)(Z) \\ &= (Z \cdot X)A \wedge B - X \wedge \{(Z \cdot A)B + (Z \cdot B)A\} \\ &= (Z \cdot X)A \wedge B - X \wedge \{(A \wedge B)(Z)\}. \end{aligned}$$

□

For a Magnus expansion θ , we denote the degree k -part of $\ell^\theta(x)$ and $L^\theta(x)$ by $\ell_k^\theta(x)$ and $L_k^\theta(x)$ respectively ($x \in \pi$).

Proposition 2.2. Let θ be a Magnus expansion. For $\forall a \in \pi$ and $\forall u \in \mathcal{L}_2$,

- (1) $L_2^\theta(a)(u) = -|a| \wedge (u(|a|))$,
- (2) $L_2^\theta(a)^2(u) = 0$.

Proof. For $a \in \pi$, $L_2^\theta(a) = \ell_1^\theta(a)\ell_1^\theta(a) = |a||a|$.

(1) For $X \wedge Y \in H \wedge H$ we have

$$\begin{aligned} L_2(a)(X \wedge Y) &= |a||a|(XY - YX) \\ &= (X \cdot |a|)|a| \wedge Y - (Y \cdot |a|)|a| \wedge X \\ &= |a| \wedge \{(X \wedge Y)(|a|)\}. \end{aligned}$$

(2) For $\forall u \in \mathcal{L}_2$, putting $Y := u(|a|)$, we have

$$L_2^\theta(a)^2(u) = |a||a|(|a| \wedge Y) = 0 + |a|(Y \cdot |a|)|a| - (Y \cdot |a|)|a||a| - 0 = 0.$$

□

When θ is an group-like expansion, the map ℓ_2^θ sends $\forall x \in \pi$ into $\mathcal{L}_2 = H \wedge H$.

Proposition 2.3 ([3], Lemma 6.4.1.). For a group-like expansion θ and for $a \in \pi$,

$$L_3^\theta(a) = |a| \wedge \ell_2^\theta(a) \in \wedge^3 H.$$

From the proposition above and Proposition 2.1 we can compute the value $L_3^\theta(a)(X)$ for $X \in H$ as

$$L_3^\theta(a)(X) = (X \cdot |a|)\ell_2^\theta(a) - |c| \wedge \{\ell_2^\theta(a)(X)\}.$$

For $\phi \in \text{Aut}(\pi)$, the map defined by

$$\tau^\theta(\phi) := T^\theta(\phi) \circ |\phi|^{-1}$$

is called the Johnson map of ϕ with respect to a Magnus expansion θ . We denote by $\tau_k^\theta(\phi)(\gamma)$ the $(k+1)$ -th degree part of $\tau^\theta(\phi)(\gamma)$ for $\gamma \in \pi$, namely

$$\tau^\theta(\phi)|_H = \text{id}_H + \sum_{k=1}^{\infty} \tau_k^\theta(\phi).$$

The following proposition proven in [3] is deduced from (2.2).

Proposition 2.4 ([3], Theorem 6.5.3.). For a non-separating simple closed curve α on Σ and a symplectic expansion θ ,

$$\tau_1^\theta(t_\alpha) = -L_3^\theta(\alpha).$$

2.4. The map ℓ_2^θ . Let θ be a Magnus expansion. We denote by $\theta_2(x)$ the degree 2 part of $\theta(x)$ for $x \in \pi$.

Proposition 2.5. For $\forall a \in \pi$, $\ell_2^\theta(a) = \theta_2(a) - \frac{1}{2}|a|^2$.

Proof. Module \widehat{T}_3 ,

$$\log(\theta(a)) \equiv \log(1 + |a| + \theta_2(a)) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (|a| + \theta_2(a))^k \equiv |a| + \theta_2(a) - \frac{1}{2}|a|^2.$$

□

It is easy to see that $\ell_2^\theta = \theta_2$ on $[\pi, \pi]$, especially $\ell_2^\theta(1) = \theta_2(1) = 0$.

Proposition 2.6. For $\forall a, b \in \pi$,

- (1) $\ell_2^\theta(ab) = \ell_2^\theta(a) + \ell_2^\theta(b) + \frac{1}{2}|a| \wedge |b|$,
- (2) $\ell_2^\theta(a^{-1}) = -\ell_2^\theta(a)$,
- (3) $\ell_2^\theta(aba^{-1}) = \ell_2^\theta(b) + |a| \wedge |b|$,
- (4) $\ell_2^\theta([a, b]) = |a| \wedge |b|$.

Proof. (1) Module \widehat{T}_3 ,

$$\theta(ab) \equiv (1 + |a| + \theta_2(a))(1 + |b| + \theta_2(b)) \equiv 1 + |a| + |b| + |a||b| + \theta_2(a) + \theta_2(b).$$

Thus $\theta_2(ab) = |a||b| + \theta_2(a) + \theta_2(b)$. By using Proposition 2.5,

$$\begin{aligned} \ell_2^\theta(ab) &= |a||b| + \theta_2(a) + \theta_2(b) - \frac{1}{2}(|a|^2 + |a||b| + |b||a| + |b|^2) \\ &= \ell_2^\theta(a) + \ell_2^\theta(b) + \frac{1}{2}(|a||b| - |b||a|). \end{aligned}$$

(2) Using (1), we have $0 = \ell_2^\theta(aa^{-1}) = \ell_2^\theta(a) + \ell_2^\theta(a^{-1})$.

(3) Using (1) and (2),

$$\begin{aligned} \ell_2^\theta(aba^{-1}) &= \ell_2^\theta(a) + \ell_2^\theta(b) + \ell_2^\theta(a^{-1}) + \frac{1}{2}|b| \wedge |a^{-1}| + \frac{1}{2}|a| \wedge |b| \\ &= \ell_2^\theta(b) + |a| \wedge |b|. \end{aligned}$$

(4) Using (1) and (3),

$$\ell_2^\theta([a, b]) = \ell_2^\theta(aba^{-1}) + \ell_2^\theta(b^{-1}) = \ell_2^\theta(b) + |a| \wedge |b| - \ell_2^\theta(b) = |a| \wedge |b|.$$

□

Note that we can define the map ℓ_2^θ from Proposition 2.6 (1) and the initial data, $\ell_2^\theta(1) = 0$ and its values on generators of π . The map ℓ mentioned in §1.1 is nothing but the map ℓ_2^θ with respect to a specific symplectic expansion θ (See §3.3).

Now let θ be a symplectic expansion.

Proposition 2.7. If $c \in \pi$ has a representative which is freely homotopic to a non-separating simple closed curve on Σ , then $\ell_2^\theta(c)(|c|) \in \mathbb{Q}|c| \subset H$.

Proof. It has shown in [3, Proposition 6.5j.1] that $L_2^\theta(c)L_3^\theta(c) = 0$ on H . On the other hand, for any $X \in H$, putting $Y := \ell_2^\theta(c)(X)$, we can compute

$$\begin{aligned} L_2^\theta(c)L_3^\theta(c)(X) &= L_2^\theta(c)((X \cdot |c|)\ell_2^\theta(c) - |c| \wedge Y) \\ &= -(X \cdot |c|)|c| \wedge \{\ell_2^\theta(c)(|c|)\} + |c| \wedge \{(|c| \wedge Y)(|c|)\} \\ &= -(X \cdot |c|)|c| \wedge \{\ell_2^\theta(c)(|c|)\}. \end{aligned}$$

Hence we obtain $|c| \wedge \{\ell_2^\theta(c)(|c|)\} = 0$. This proves the proposition. □

The action of $\omega = \sum_{i=1}^g X_i \wedge Y_i \in H \wedge H$ on H is as follows: For $A = \sum_{j=1}^g \xi_j X_j + \eta_j Y_j \in H$, $(\xi_j, \eta_j \in \mathbb{Q})$,

$$\omega(A) = \sum_{i=1}^g (A \cdot X_i) Y_i - (A \cdot Y_i) X_i = \sum_{i=1}^g -\eta_i Y_i - \xi_i X_i = -A.$$

Since θ is a symplectic expansion, i.e., $\ell_2^\theta(\zeta) = \omega$, we have the following:

Proposition 2.8. *For the boundary curve ζ of $\Sigma_{g,1}$ and $\forall a \in \pi$, $\ell_2^\theta(\zeta)(|a|) = -|a|$.*

3. PROOF OF THEOREM 1.1

Let α, β be oriented simple closed curves on Σ . For the reason mentioned in Remark 1.1(1), we may assume that α is non-separating on Σ . We choose based loops a and b with the base point $p \in \partial\Sigma$ which are freely homotopic to α and β respectively, and we will denote the based homotopy classes of them by the same symbols a, b .

Here we recall the classical formula about the action of t_α , a right-handed Dehn twist along α , on H :

$$(3.1) \quad |t_\alpha|(X) = X + (|a| \cdot X)|a|, \quad (\forall X \in H).$$

3.1. Degree two part of symplectic expansion of $t_\alpha(b)$. Focusing on the $H^{\otimes 2}$ -part of symplectic expansion of $t_\alpha(b)$ and of b we have the following.

Lemma 3.1. *Curves α, β and $a, b \in \pi$ as above. Let θ be a symplectic expansion of π . Suppose that $i_A(\alpha, \beta) = 0$. Then*

$$\theta_2(t_\alpha(b)) - \theta_2(b) = |a| \wedge (\ell_2^\theta(a)|b| + \ell_2^\theta(b)|a|).$$

Proof. The expansion of $t_\alpha(b)$ by a symplectic expansion θ of π modulo \widehat{T}_3 is computed as

$$(3.2) \quad \begin{aligned} \theta(t_\alpha(b)) &= T^\theta(t_\alpha)(\theta(b)) \\ &\equiv T^\theta(t_\alpha)(1 + |b| + \theta_2(b)) \mod \widehat{T}_3 \\ &= 1 + T^\theta(t_\alpha)(|b|) + T^\theta(t_\alpha)(\theta_2(b)). \end{aligned}$$

Claim 1. $T^\theta(t_\alpha)(|b|) \equiv |b| + |a| \wedge \{\ell_2^\theta(a)(|b|)\} \mod \widehat{T}_3$

Proof of Claim 1. It follows from the classical formula (3.1) and the condition $|a| \cdot |b| = 0$ that

$$|t_\alpha(b)| = |b|.$$

Recall the definition of the Johnson map:

$$T^\theta(t_\alpha) = \tau^\theta(t_\alpha) \circ |t_\alpha|.$$

Then we have

$$(3.3) \quad T^\theta(t_\alpha)(|b|) = \tau^\theta(t_\alpha)(|b|) \equiv |b| + \tau_1^\theta(t_\alpha)(|b|) \mod \widehat{T}_3.$$

Applying Proposition 2.4 (note that we assume α is non-separating loop), 2.3 and 2.1(2),

$$\tau_1^\theta(t_\alpha)(|b|) = -L_3^\theta(\alpha)(|b|) = -(|a| \wedge \ell_2^\theta(a))(|b|) = |a| \wedge \{\ell_2^\theta(a)(|b|)\}.$$

□

Claim 2. $T^\theta(t_\alpha)(\theta_2(b)) \equiv \theta_2(b) + |a| \wedge \{\ell_2^\theta(b)(|a|)\} \mod \widehat{T}_3$

Proof of Claim 2. Note that the map $|t_\alpha| \in \text{Hom}(H, \widehat{T}_1)$ is also regarded as an automorphism of \widehat{T} such that $|t_\alpha|(X_1 \cdots X_p) = |t_\alpha|^{\otimes p}(X_1 \cdots X_p) = |t_\alpha|(X_1) \cdots |t_\alpha|(X_p)$ for $X_1 \cdots X_p \in H^{\otimes p}$. Thus we have

$$T^\theta(t_\alpha)(\theta_2(b)) = \tau^\theta(t_\alpha)(|t_\alpha|^{\otimes 2}\theta_2(b))$$

The lowest degree part of $\tau^\theta(t_\alpha)(u)$ for $u \in H^{\otimes 2}$ is equal to u since that of $\tau^\theta(t_\alpha)(X)$ for $X \in H$ is X as we can see in (3.3). Applying Proposition 2.5,

$$\begin{aligned} \tau^\theta(t_\alpha)(|t_\alpha|^{\otimes 2}\theta_2(b)) &\equiv |t_\alpha|^{\otimes 2}\theta_2(b) \pmod{\widehat{T}_3} \\ &= |t_\alpha|^{\otimes 2}\ell_2^\theta(b) + \frac{1}{2}|t_\alpha|^{\otimes 2}|b|^2. \end{aligned}$$

Now we claim that $\forall u \in \mathcal{L}_2$, $|t_\alpha|^{\otimes 2}(u) = u + |a| \wedge \{u(|a|)\}$. In fact, for $X \wedge Y \in \mathcal{L}_2$,

$$\begin{aligned} |t_\alpha|^{\otimes 2}(X \wedge Y) &= (X + (|a| \cdot X)|a|) \wedge (Y + (|a| \cdot Y)|a|) \\ &= X \wedge Y + |a| \wedge \{(X \wedge Y)(|a|)\}. \end{aligned}$$

We then have

$$\tau^\theta(t_\alpha)(|t_\alpha|^{\otimes 2}\theta_2(b)) = \ell_2^\theta(b) + |a| \wedge \{\ell_2^\theta(b)(|a|)\} + \frac{1}{2}|b|^2 = \theta_2(b) + |a| \wedge \{\ell_2^\theta(b)(|a|)\}.$$

□

From (3.2) with Claim 1 and Claim 2 we obtain

$$\theta(t_\alpha(b)) \equiv 1 + |b| + \theta_2(b) + |a| \wedge \{\ell_2^\theta(a)(|b|) + \ell_2^\theta(b)(|a|)\} \pmod{\widehat{T}_3},$$

and a proof of the lemma has done. □

3.2. The value $\ell_2^\theta(a)(|b|) + \ell_2^\theta(b)(|a|)$. When $i_G(\alpha, \beta) = 0$ holds, there exists a based homotopy class $b_0 \in \pi$ corresponding to the curve β so that the Dehn twist t_α fixes it. Then $\theta(t_\alpha(b_0)) = \theta(b_0)$, especially $\theta_2(t_\alpha(b_0)) - \theta_2(b_0) = 0$. According to the lemma above, if we take a symplectic expansion θ , we can restate the last equation that $|a| \wedge \{\ell_2^\theta(a)(|b_0|) + \ell_2^\theta(b_0)(|a|)\} = 0$.

In this subsection, we discuss the behavior of the value $\ell_2^\theta(a)(|b|) + \ell_2^\theta(b)(|a|)$ for arbitrary chosen homotopy classes $|a|$ and $|b|$ of the curves. In the case where the two curves α and β are both non-separating on Σ , we first suppose that corresponding homotopy classes $|a|$ and $|b|$ of the curves are linearly independent on H , and show that the value $\ell_2^\theta(a)(|b|) + \ell_2^\theta(b)(|a|)$ with any symplectic expansion θ is in $\mathbb{Z}|a| + \mathbb{Z}|b| \subset H$ if $i_G(\alpha, \beta) = 0$. Second, we see that, when $|a|$ and $|b|$ are linearly dependent on H , $\ell_2^\theta(a)(|b|) + \ell_2^\theta(b)(|a|)$ with a certain type of symplectic expansion θ is always in $\mathbb{Z}|a| + \mathbb{Z}|b| (= \mathbb{Z}|a|)$. Finally, in the case where β is separating, we will see the same behavior of the value $\ell_2^\theta(a)(|b|) + \ell_2^\theta(b)(|a|) = \ell_2^\theta(b)(|a|)$ as in the first case.

3.2.1. The case where β is non-separating. Let α, β be both non-separating simple closed curves on Σ . We take homotopy classes a and b corresponding to the curves arbitrarily.

In the case where $|a|$ and $|b|$ are linearly independent on H , the following holds:

Theorem 3.2. *Curves α, β and $a, b \in \pi$ as above. Suppose that $|a|$ and $|b|$ are linearly independent on H . Let θ be a symplectic expansion. If $i_G(\alpha, \beta) = 0$, then $\ell_2^\theta(a)(|b|) + \ell_2^\theta(b)(|a|) \in \mathbb{Z}|a| + \mathbb{Z}|b|$.*

Proof. Suppose that the curves α, β satisfy $i_G(\alpha, \beta) = 0$. While it is not always true that $t_\beta(a) = a$ and $t_\alpha(b) = b$, we may take based loops a_0 and b_0 with the base point $p \in \partial\Sigma$ such that they are freely homotopic to α and β respectively and satisfy

$$t_\beta(a_0) = a_0, \quad t_\alpha(b_0) = b_0,$$

because α and β are non-separating loops. Note that $|a_0| = |a|$ and $|b_0| = |b|$. Taking a symplectic expansion θ and applying Lemma 3.1, we have

$$\theta_2(t_\beta(a_0)) - \theta_2(a_0) = |b| \wedge \{\ell_2^\theta(a_0)(|b|) + \ell_2^\theta(b_0)(|a|)\} = 0$$

and

$$\theta_2(t_\alpha(b_0)) - \theta_2(b_0) = |a| \wedge \{\ell_2^\theta(a_0)(|b|) + \ell_2^\theta(b_0)(|a|)\} = 0.$$

It follows from these two equations that

$$\ell_2^\theta(a_0)(|b|) + \ell_2^\theta(b_0)(|a|) \in \mathbb{Q}|a| \cap \mathbb{Q}|b|.$$

Since $|a|$ and $|b|$ are linearly independent, $\mathbb{Q}|a| \cap \mathbb{Q}|b| = \{0\}$. We then know that

$$\ell_2^\theta(a_0)(|b|) + \ell_2^\theta(b_0)(|a|) = 0.$$

There exist $c, d \in \pi$ such that $a_0 = cac^{-1}$, $b_0 = dbd^{-1}$. By using Proposition 2.6(3) and $|a| \cdot |b| = 0$,

$$\begin{aligned} \ell_2^\theta(a_0)(|b|) + \ell_2^\theta(b_0)(|a|) &= \ell_2^\theta(cac^{-1})(|b|) + \ell_2^\theta(dbd^{-1})(|a|) \\ &= \{\ell_2^\theta(a) + |c| \wedge |a|\}(|b|) + \{\ell_2^\theta(b) + |d| \wedge |b|\}(|a|) \\ &= \ell_2^\theta(a)(|b|) + \ell_2^\theta(b)(|a|) + (|b| \cdot |c|)|a| + (|a| \cdot |d|)|b|. \end{aligned}$$

Thus we conclude that $\ell_2^\theta(a)(|b|) + \ell_2^\theta(b)(|a|) \in \mathbb{Z}|a| + \mathbb{Z}|b|$. \square

Next we suppose that $|a|$ and $|b|$ are linearly dependent, i.e., $|b| = k|a|$, ($k \in \mathbb{Q}$). We claim that $k = \pm 1$. In fact, since β is non-separating, we can take a symplectic generators $\{x_1, y_1, \dots, x_g, y_g\}$ of π such that $b = x_1$, and then we have $1 = X_1 \cdot Y_1 = |b| \cdot Y_1 = k|a| \cdot Y_1$, which proves the claim.

In this case we choose a symplectic expansion θ such that $\ell_2^\theta(\pi) \subset \frac{1}{2}\wedge^2 H_{\mathbb{Z}} \subset \mathcal{L}_2$, where $H_{\mathbb{Z}} = H_1(\Sigma; \mathbb{Z})$. There exists such a symplectic expansion (see §3.3). We have the following:

Theorem 3.3. *Curves α, β , homology classes a, b as above. Suppose that $|b| = \pm|a|$. If we take a symplectic expansion θ such that $\ell_2^\theta(\pi) \subset \frac{1}{2}\wedge^2 H_{\mathbb{Z}}$, then it is always true that $\ell_2^\theta(a)(|b|) + \ell_2^\theta(b)(|a|) \in \mathbb{Z}|a|$.*

Proof. There exists $d \in [\pi, \pi]$ such that $b = a^\epsilon d$, where $\epsilon = \pm 1$.

$$\ell_2^\theta(a)(|b|) + \ell_2^\theta(b)(|a|) = \ell_2^\theta(a)(\epsilon|a|) + \ell_2^\theta(a^\epsilon d)(|a|) = 2\epsilon\ell_2^\theta(a)(|a|) + \ell_2^\theta(d)(|a|).$$

It follows from the condition $\ell_2^\theta(\pi) \subset \frac{1}{2}\wedge^2 H_{\mathbb{Z}}$ and Proposition 2.7 that $2\epsilon\ell_2^\theta(a)(|a|) \in \mathbb{Z}|a|$. Since α and β are non-separating, we know from Proposition 2.7 that $\ell_2^\theta(a)(|a|)$ and $\ell_2^\theta(b)(|b|)$ are both in $\mathbb{Q}|a| = \mathbb{Q}|b|$. We can compute $\ell_2^\theta(b)(|b|)$ as

$$\ell_2^\theta(b)(|b|) = \ell_2^\theta(a^\epsilon d)(\epsilon|a|) = \ell_2^\theta(a)(|a|) + \epsilon\ell_2^\theta(d)(|a|).$$

It follows that $\ell_2^\theta(d)(|a|) \in \mathbb{Q}|a|$. Since $d \in [\pi, \pi]$, we have $\ell_2^\theta(d) \in H_{\mathbb{Z}} \wedge H_{\mathbb{Z}}$ by Proposition 2.6(4), and it indicates $\ell_2^\theta(d)(|a|) \in H_{\mathbb{Z}}$. Therefore we obtain $\ell_2^\theta(d)(|a|) \in \mathbb{Z}|a|$ and the proof is done. \square

3.2.2. *The case where β is separating.*

Theorem 3.4. *Let α be a non-separating simple closed curve and β a separating simple closed curve on Σ . Homotopy classes a and b for the curves as above. Let θ be a symplectic expansion. If $i_G(\alpha, \beta) = 0$, then $\ell_2^\theta(b)(|a|) \in \mathbb{Z}|a|$.*

Proof. Suppose $i_G(\alpha, \beta) = 0$ and take a based loop b_0 satisfying $t_\alpha(b_0) = b_0$. Note that $|b_0| = |b| = 0$. Taking a symplectic expansion θ and applying Lemma 3.1, we have

$$\theta_2(t_\alpha(b_0)) - \theta_2(b_0) = |a| \wedge \{\ell_2^\theta(b_0)(|a|)\} = 0.$$

It follows that $\ell_2^\theta(b_0)(|a|) \in \mathbb{Q}|a|$. There exist $d \in \pi$ such that $b_0 = dbd^{-1}$. From Proposition 2.6(3) and $|b| = 0$,

$$\ell_2^\theta(b_0)(|a|) = \ell_2^\theta(dbd^{-1})(|a|) = \ell_2^\theta(b)(|a|).$$

Thus we have $\ell_2^\theta(b)(|a|) \in \mathbb{Q}|a|$. As in the proof of the previous theorem we can see $\ell_2^\theta(b)(|a|) \in H_{\mathbb{Z}}$. This completes the proof. \square

3.3. The map $\ell : \pi \rightarrow H \wedge H$. With respect to a symplectic generators $\{x_j, y_j\}_{1 \leq j \leq g}$ of $\pi = \pi_1(\Sigma_{g,1}, p)$ ($g \geq 1$), it is shown [5, 4] that there exists a symplectic expansion θ_0 of π satisfying

$$\ell_2^{\theta_0}(x_j) = \frac{1}{2}X_j \wedge Y_j, \quad \ell_2^{\theta_0}(y_j) = -\frac{1}{2}X_j \wedge Y_j,$$

for $1 \leq j \leq g$. We can easily see from Proposition 2.6 that the map $\ell_2^{\theta_0}$ is identical with the map ℓ defined in §1. Note that $\ell(\pi) \subset \frac{1}{2}\wedge^2 H_{\mathbb{Z}}$. Applying Theorem 3.2, 3.3 and 3.4 to the symplectic expansion θ_0 , we obtain Theorem 1.1.

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